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Abstract

An approximation Ansatz for the operator solution, $U(z', z)$, of a hyperbolic first-order pseudodifferential equation, $\partial_z + a(z, x, D_x)$ with $\mathrm{Re}(a) \geq 0$, is constructed as the composition of global Fourier integral operators with complex phases. The symbol $a(z, .)$ is assumed to have a regularity as low as Hölder, $\mathscr{C}^{0,\alpha}$, with respect to the evolution parameter z. We prove a convergence result for the Ansatz to $U(z', z)$ in some Sobolev space as the number of operators in the composition goes to ∞, with a convergence of order α. We also study the consequences of some truncation approximations of the symbol $a(z, .)$ in the construction of the Ansatz.

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Introduction

We consider the Cauchy problem

$$\partial_z u + a(z, x, D_x)u = 0, \quad 0 < z \leq Z \tag{1}$$

$$u\,|_{z=0} = u_0, \tag{2}$$

with $Z > 0$ and $a(z, x, \xi)$ continuous with respect to (w.r.t.) z with values in $S^1(\mathbb{R}^n \times \mathbb{R}^n)$ with the usual notation $D_x = \frac{1}{i}\partial_x$. Further assumptions will be made on the symbol $a(z, x, \xi)$. We denote $U(z, 0)$ the solution operator of (1)–(2).

This article concerns a representation of the solution of (1)–(2), and more generally of $U(z, 0)$, as a limit of compositions of infinitesimal approximations as introduced in [12] and analyses its convergence rate. Here, we prove an optimal convergence rate. Such a representation is possible even in the case of a regularity as low as Hölder with respect to the evolution parameter z, in which case the classical Fourier-integral-operator parametrix construction is not feasible [3, 17]. As opposed to the classical parametrix construction we obtain here an exact representation of the solution of (1)–(2).

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When $a(z,x,\xi)$ is independent of x and z it is natural to treat such a problem by means of Fourier transformation:

$$u(z,x') = \iint \exp[i\langle x' - x|\xi\rangle - za(\xi)]\, u_0(x)\, đ\xi\, dx,$$

where $đ\xi := d\xi/(2\pi)^n$. For this to be well defined for all $u_0 \in \mathscr{S}(\mathbb{R}^n)$ we shall impose the real part of the principal symbol of a to be non-negative. When the symbol a depends on both x and z we can naively expect

$$u(z,x') \approx u_1(z,x') := \iint \exp[i\langle x' - x|\xi\rangle - za(0,x',\xi)]\, u_0(x)\, đ\xi\, dx$$

for z small and hence approximately solve the Cauchy problem (1)–(2) for $z \in [0, z^{(1)}]$ with $z^{(1)}$ small. If we want to progress in the z direction we have to solve the Cauchy problem

$$\partial_z u + a(z,x,D_x)u = 0, \quad z^{(1)} < z \leq Z,$$
$$u(z,.)\,|_{z=z^{(1)}} = u_1(z^{(1)},.),$$

which we again approximately solve by

$$u(z,x') \approx u_2(z,x') := \iint \exp[i\langle x' - x|\xi\rangle - (z - z^{(1)})a(z^{(1)},x',\xi)]\, u_1(z^{(1)},x)\, đ\xi\, dx.$$

This procedure can be iterated until we reach $z = Z$.

If we denote by $\mathcal{G}_{(z',z)}$ the operator with kernel

$$G_{(z',z)}(x',x) = \int \exp[i\langle x' - x|\xi\rangle] \exp[-(z'-z)a(z,x',\xi)]\, đ\xi,$$

then combining all iteration steps above involves composition of such operators: let $0 \leq z^{(1)} \leq \cdots \leq z^{(k)} \leq Z$, we then have

$$u_{k+1}(z,x) = \mathcal{G}_{(z,z^{(k)})} \circ \mathcal{G}_{(z^{(k)},z^{(k-1)})} \circ \cdots \circ \mathcal{G}_{(z^{(1)},0)}(u_0)(x),$$

when $z \geq z^{(k)}$. We then define the operator $\mathcal{W}_{\mathfrak{P},z}$ for a subdivision of $\mathfrak{P}$ of $[0,Z]$, $\mathfrak{P} = \{z^{(0)}, z^{(1)}, \ldots, z^{(N)}\}$, with $0 = z^{(0)} < z^{(1)} < \cdots < z^{(N)} = Z$,

$$\mathcal{W}_{\mathfrak{P},z} := \begin{cases} \mathcal{G}_{(z,0)} & \text{if } 0 \leq z \leq z^{(1)}, \\ \mathcal{G}_{(z,z^{(k)})} \displaystyle\prod_{i=k}^{1} \mathcal{G}_{(z^{(i)},z^{(i-1)})} & \text{if } z^{(k)} \leq z \leq z^{(k+1)}. \end{cases}$$

According to the procedure described above, $\mathcal{W}_{\mathfrak{P},z}(u_0)$ yields an approximation Ansatz for the solution to the Cauchy problem (1)–(2) with step size $\Delta_{\mathfrak{P}} = z_i - z_{i-1}$, $i = 1, \ldots, N$. The operator $\mathcal{G}_{(z',z)}$ is a Fourier integral operator (FIO) and is often referred to as the *thin-slab propagator* (see e.g. [2, 1, 12]).

Note that a similar procedure can be used to show the existence of an evolution system by approximating it by composition of semigroup solutions of the Cauchy problem with z 'frozen' in $a(z,x,D_x)$ [8, 16]. Note that the thin-slab propagator $\mathcal{G}_{(z',z)}$ is however not a semigroup nor an evolution family here (see [12, Section 3] and [15]).

The approximation Ansatz proposed here is a tool to compute approximations of the exact solution to the Cauchy problem (1)–(2) and yields a representation of these solution by means of infinite products of FIOs, even in the case of *low regularity* on the symbol $a(z, x, \xi)$ w.r.t. z. Such computations in applications to geophysical problems have been used in [2]. In exploration seismology one is confronted with solving equations of the type

$$(\partial_z - ib(z, x, D_t, D_x) + c(z, x, D_t, D_x))v = 0, \tag{3}$$

$$v(0, .) = v_0(.), \tag{4}$$

where t is time, z is the vertical coordinate and x is the lateral or transverse coordinate. The operators b and c are of first order, with real principal parts, b_1 and c_1, where $c_1(z, x, \tau, \xi)$ is non-negative. Note that the Cauchy problem (1)–(2) studied here is more general. In seismology, low regularity in the coefficients is often encountered. Because of very fine bedding in sedimentary rocks, for example, heterogeneities can be observed at every scale. Here, we shall consider a regularity as low as Hölder in the coefficients w.r.t. the evolution parameter z.

The Cauchy problem (3)–(4) is obtained by a (microlocal) decoupling of the up-going and down-going wavefields in the acoustic wave equation (see Appendix A in [12] and [18] for details). In practice, the proposed Ansatz can then be a tool to approximate the exact solution for the purpose of imaging the Earth's interior [2, 1]. For such applications one is inclined to approximate the symbol $a(z, x, \xi)$ itself. We shall study some consequences of some type of approximations.

In the present paper, we complete the analysis of the convergence of the approximation scheme $\mathcal{W}_{\mathfrak{P}}$ in Sobolev spaces developed in [12]. Section 1 introduces the Cauchy problem we study and the precise assumptions made on the symbol $a(z, x, \xi)$, especially on the real part, c_1, and imaginary part, $-b_1$, of its principal symbol. In Section 2 we study the convergence of the Ansatz $\mathcal{W}_{\mathfrak{P},z}(u_0)$ to the solution of the Cauchy problem (1)–(2) in Sobolev spaces as $\Delta_{\mathfrak{P}}$ goes to 0. A convergence in norm of $\mathcal{W}_{\mathfrak{P},z}$ to the solution operator of the Cauchy problem (1)–(2) is actually obtained (Theorem 2.10):

$$\lim_{\Delta_{\mathfrak{P}} \to 0} \|\mathcal{W}_{\mathfrak{P},z} - U(z, 0)\|_{(H^{(s+1)}, H^{(s)})} = 0,$$

with a convergence rate of order α when $a(z, .)$ is in $\mathscr{C}^{0,\alpha}$ w.r.t. z, $\alpha > 0$ thus improving the result in [12], where only a convergence of order $\frac{1}{2}$ was proven in the case $\alpha \geq \frac{1}{2}$. Observe that the proposed Ansatz corresponds to a first-order approximation. The convergence rate found here, of order one in the case of Lipschitz coefficients, is thus optimal. This is in agreement with the convergence rate for the wavefront set of the Ansatz proven in [15].

We furthermore obtain (Theorem 2.10)

$$\lim_{\Delta_{\mathfrak{P}} \to 0} \|\mathcal{W}_{\mathfrak{P},z} - U(z, 0)\|_{(H^{(s+1)}, H^{(s+r)})} = 0, \quad 0 \leq r < 1$$

with a convergence rate of order $\alpha(1 - r)$ while the operator $\mathcal{W}_{\mathfrak{P},z}$ strongly converges to $U(z, 0)$ in $H^{(s+1)}$. The proof relies on the analysis of the thin-slab propagator $\mathcal{G}_{(z',z)}$ in [12].

At the end of Section 2 we relax some regularity property of the symbol $a(z, .)$ w.r.t. z by the introduction of another, yet natural, Ansatz: following [11], the thin-slab propagator, $\mathcal{G}_{(z',z)}$, is replaced by the operator $\widehat{\mathcal{G}}_{(z',z)}$ with kernel

$$\widehat{G}_{(z',z)}(x', x) = \int \exp[i\langle x' - x|\xi\rangle] \exp[-\textstyle\int_z^{z'} a(s, x', \xi) ds] \, d\!\!\!^{-}\xi.$$

In Section 3, we study the implications of approximating the symbol $a(z,x,\xi)$ in the convergence result. Such approximations are of importance in practice, for instance in geophysics, as the symbol $a(z,x,\xi)$ is often known by an asymptotic series, which is truncated for computational reasons. If the symbol $a(z,.)$ is given as

$$a(z,x,\xi) \sim \sum_{j=0}^{\infty} a_{1-j}(z,x,\xi),$$

one may truncate this series and, instead, use

$$\underline{a}(z,x,\xi) = \sum_{j=0}^{k} a_{1-j}(z,x,\xi)$$

in the definition of the Ansatz $\mathcal{W}_{\mathfrak{P},z}$. More generally, we may assume that $a - \underline{a} \in \mathscr{C}^{0,\alpha}([0,Z], S^{-k}(\mathbb{R}^n \times \mathbb{R}^n))$. If we denote by $\underline{\mathcal{W}}_{\mathfrak{P},z}$ the new associated Ansatz, we shall see in fact that $\mathcal{W}_{\mathfrak{P},z}(u_0) - \underline{\mathcal{W}}_{\mathfrak{P},z}(u_0)$ is in $H^{(s+k)}(\mathbb{R}^n)$ and moreover

$$\|\mathcal{W}_{\mathfrak{P},z} - \underline{\mathcal{W}}_{\mathfrak{P},z}\|_{(H^{(s)},H^{(s+k)})} \leq CZ \sup_{z\in[0,Z]} p(a(z,.) - \underline{a}(z,.))\ \exp[CZ], \qquad z \in [0,Z],$$

for some appropriately chosen seminorm p on $S^{-k}(\mathbb{R}^n \times \mathbb{R}^n)$.

In this paper, we shall use the notations of [12]. When the constant C is used, its value may change from one line to another. If we want to keep track of the value of a constant we shall use another letter. When we write that a function is bounded w.r.t. z and/or Δ we shall actually mean that z is to be taken in the interval $[0,Z]$ and Δ in some interval $[0,\Delta_{max}]$ unless otherwise stipulated. We shall generally write X, X', Y, ... for $\mathbb{R}^n$, according to variables, e.g., x, x', y, ...

Throughout the paper, we use spaces of global symbols; a function $a \in \mathscr{C}^\infty(\mathbb{R}^n \times \mathbb{R}^p)$ is in $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^p)$, $0 < \rho \leq 1$, $0 \leq \delta < 1$, if for all multi-indices α, β there exists $C_{\alpha\beta} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta}\,(1+|\xi|)^{m-\rho|\beta|+\delta|\alpha|}, \quad x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^p.$$

The best possible constants $C_{\alpha\beta}$, i.e.,

$$p_{\alpha\beta}(a) := \sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^p} (1+|\xi|)^{-m+\rho|\beta|-\delta|\alpha|} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)|, \tag{5}$$

define seminorms for a Fréchet space structure on $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^p)$. As usual we write $S^m_\rho(\mathbb{R}^n \times \mathbb{R}^p)$ in the case $\rho = 1-\delta$, $\frac{1}{2} \leq \rho < 1$, and $S^m(\mathbb{R}^n \times \mathbb{R}^p)$ in the case $\rho = 1$, $\delta = 0$.

We shall use, in a standard way, the notation # for the composition of symbols of pseudodifferential operators (ψDO) and make use of the oscillatory integral representation of the resulting symbol.

1 The homogeneous first-order hyperbolic equation

Let $s \in \mathbb{R}$ and $Z > 0$. We consider the Cauchy problem

$$\partial_z u + a(z,x,D_x)u = 0, \quad 0 < z \leq Z, \tag{1.1}$$

$$u\,|_{z=0} = u_0 \in H^{(s+1)}(\mathbb{R}^n), \tag{1.2}$$

where the symbol $a(z,x,\xi)$ satisfies the following assumption

Assumption 1.1.

$$a_z(x,\xi) = a(z,x,\xi) = -i\,b(z,x,\xi) + c(z,x,\xi),$$

where $b \in \mathscr{C}^0([0,Z], S^1(\mathbb{R}^n \times \mathbb{R}^n))$, with real principal symbol b_1 homogeneous of degree 1 for $|\xi|$ large enough and $c \in \mathscr{C}^0([0,Z], S^1(\mathbb{R}^n \times \mathbb{R}^n))$ with non-negative principal symbol c_1 homogeneous of degree 1 for $|\xi|$ large enough. Without loss of generality we can assume that b_1 and c_1 are homogeneous of degree 1 for $|\xi| \geq 1$.

In Section 2, we shall make further regularity assumptions, w.r.t. z, on the symbol $a(z,x,\xi)$.

We denote by $a_1 = -ib_1 + c_1$ the principal symbol of a and write $b = b_1 + b_0$ with $b_0 \in \mathscr{C}^0([0,Z], S^0(\mathbb{R}^n \times \mathbb{R}^n))$ and $c = c_1 + c_0$ with $c_0 \in \mathscr{C}^0([0,Z], S^0(\mathbb{R}^n \times \mathbb{R}^n))$. Assumption 1.1 ensures that the hypotheses (i)–(iii) of Theorem 23.1.2 in [5] are satisfied. Then there exists a unique solution in $\mathscr{C}^0([0,Z], H^{(s+1)}(\mathbb{R}^n)) \cap \mathscr{C}^1([0,Z], H^{(s)}(\mathbb{R}^n))$ to the Cauchy problem (1.1)–(1.2).

Furthermore, we have the following energy estimate [5, Lemma 23.1.1] for any function in $\mathscr{C}^1([0,Z], H^{(s)}(\mathbb{R}^n)) \cap \mathscr{C}^0([0,Z], H^{(s+1)}(\mathbb{R}^n))$

$$(1.3) \qquad \sup_{z\in[0,Z]} \exp[-\lambda z]\, \|u(z,.)\|_{H^{(s)}} \leq \|u(0,.)\|_{H^{(s)}} + 2\int_0^Z \exp[-\lambda z]\, \|\partial_z u + a_z(x,D_x)u\|_{H^{(s)}} dz,$$

with λ large enough (λ solely depending on s).

By Proposition 9.3 in [4, Chapter VI] the family of operators $(a_z)_{z\in[0,Z]}$ generates a strongly continuous evolution system. Let $U(z',z)$ denote the corresponding evolution system:

$$U(z'',z') \circ U(z',z) = U(z'',z), \quad Z \geq z'' \geq z' \geq z \geq 0.$$

with

$$\partial_z U(z,z_0)u_0 + a(z,x,D_x)U(z,z_0)u_0 = 0, \quad 0 \leq z_0 < z \leq Z,$$
$$U(z_0,z_0)u_0 = u_0 \in H^{(s+1)}(\mathbb{R}^n)$$

while $U(z,z_0)u_0 \in H^{(s+1)}(\mathbb{R}^n)$ for all $z \in [z_0,Z]$. For the Cauchy problem (1.1)–(1.2) we take $z_0 = 0$.

We now recall some results obtained in [12]. Let $z',z \in [0,Z]$ with $z' \geq z$ and let $\Delta := z' - z$. Define $\phi_{(z',z)} \in \mathscr{C}^\infty(X' \times X \times \mathbb{R}^n)$ by

$$(1.4) \quad \phi_{(z',z)}(x',x,\xi) := \langle x' - x|\xi\rangle + i\Delta a_1(z,x',\xi) = \langle x' - x|\xi\rangle + \Delta b_1(z,x',\xi) + i\Delta c_1(z,x',\xi).$$

Lemma 1.2. *$\phi_{(z',z)}$ is a non-degenerate complex phase function of positive type (at any point (x_0',x_0,ξ_0) where $\partial_\xi \phi_{(z',z)} = 0$).*

We put

$$(1.5) \qquad g_{(z',z)}(x,\xi) := \exp[-\Delta a_0(z,x,\xi)] \in S^0(X \times \mathbb{R}^n)$$

and define a distribution kernel $G_{(z',z)}(x',x) \in \mathscr{D}'(X' \times X)$ by the oscillatory integral

$$G_{(z',z)}(x',x) = \int \exp[i\langle x'-x|\xi\rangle] \exp[-\Delta a(z,x',\xi)]\, d\!\!{}^-\xi$$
$$= \int \exp[i\phi_{(z',z)}(x',x,\xi)]\, g_{(z',z)}(x',\xi)\, d\!\!{}^-\xi.$$

We denote the associated operator by $\mathcal{G}_{(z',z)}$. (This corresponds to the thin-slab propagator (see e.g. [2, 1]).)

Let $J_{(z',z)}$ be the canonical ideal locally generated by the phase function $\phi_{(z',z)}$.

Proposition 1.3. *There exists $\Delta_1 > 0$, such that, for all $z', z \in [0, Z]$, with $z' \geq z$ and $\Delta = z' - z \leq \Delta_1$, the phase function $\phi_{(z',z)}$ globally generates the canonical ideal $J_{(z',z)}$. Alternatively, it is also generated by the functions*

$$(1.6)\qquad \begin{aligned} v_{\xi_j}(x',x,\xi',\xi) &= \partial_{x'_j}\phi_{(z',z)}(x',x,\xi) - \xi'_j = \xi_j - \xi'_j + i\Delta\partial_{x_j}a_1(z,x',\xi),\\ v_{x_j}(x',x,\xi',\xi) &= \partial_{\xi_j}\phi_{(z',z)}(x',x,\xi) = x'_j - x_j + i\Delta\partial_{\xi_j}a_1(z,x',\xi), \end{aligned}$$

$j = 1, \ldots, n$.

Proposition 1.4. *If $0 \leq \Delta = z' - z \leq \Delta_1$, with $z' \geq z$, then the operator $\mathcal{G}_{(z',z)}$ is a global Fourier integral operator with complex phase and its kernel $G_{(z',z)}$ is in $I^0(X' \times X, (J_{(z',z)})', \Omega^{1/2}_{X'\times X})$.*

We denote the half density bundle on $X' \times X$ by $\Omega^{1/2}_{X'\times X}$ and note that $(J_{(z',z)})'$ stands for the twisted canonical ideal, i.e., a Lagrangian ideal (see Section 25.5 in [6]) and $I^0(X' \times X, (J_{(z',z)})', \Omega^{1/2}_{X'\times X})$ denotes the associated Lagrangian distributions of order 0.

Theorem 1.5. *Let $s \in \mathbb{R}$. There exists $\Delta_2 > 0$ such that if $z', z \in [0, Z]$, $z \leq z'$, with $0 \leq \Delta := z' - z \leq \Delta_2$ then $\mathcal{G}_{(z',z)}$ continuously maps $\mathscr{S}$ into $\mathscr{S}$, $\mathscr{S}'$ into $\mathscr{S}'$, and $H^{(s)}(\mathbb{R}^n)$ into $H^{(s)}(\mathbb{R}^n)$. In fact, there exists $C > 0$ such that we have the following norm estimate*

$$\|\mathcal{G}_{(z',z)}\|_{(H^{(s)},H^{(s)})} \leq 1 + C\Delta,$$

uniformly w.r.t. z and z' as above.

The approximation Ansatz is defined by

Definition 1.6. *Let $\mathfrak{P} = \{z^{(0)}, z^{(1)}, \ldots, z^{(N)}\}$ be a subdivision of $[0, Z]$ with $0 = z^{(0)} < z^{(1)} < \cdots < z^{(N)} = Z$ such that $z^{(i+1)} - z^{(i)} = \Delta_{\mathfrak{P}}$. The operator $\mathcal{W}_{\mathfrak{P},z}$ is defined as*

$$\mathcal{W}_{\mathfrak{P},z} := \begin{cases} \mathcal{G}_{(z,0)} & \text{if } 0 \leq z \leq z^{(1)},\\ \mathcal{G}_{(z,z^{(k)})} \displaystyle\prod_{i=k}^{1} \mathcal{G}_{(z^{(i)},z^{(i-1)})} & \text{if } z^{(k)} \leq z \leq z^{(k+1)}. \end{cases}$$

We chose to use a constant-step subdivision of the interval $[0, Z]$ but the method and results presented here can be naturally adapted to any subdivision of $[0, Z]$.

2 Convergence result

Let $s \in \mathbb{R}$. In [12, Proposition 3.8], it was proved that, for Δ sufficiently small, we have

$$\|(\partial_{z'} + a_{z'}(x, D_x))\mathcal{G}_{(z',z)}\|_{(H^{(s)},H^{(s-1)})} \leq C\Delta^{\beta},$$

with $\beta = \min(\alpha, \frac{1}{2})$ if $a_z(x,\xi)$ is of Hölder regularity of order α, w.r.t. z. This estimate yields the convergence result in Sobolev spaces found in [12] with a rate of order β:

$$\|\mathcal{W}_{\mathfrak{P},z} - U(z,0)\|_{(H^{(s+1)},H^{(s)})} \leq C\Delta_{\mathfrak{P}}^{\beta}, \quad z \in [0,Z].$$

Here, we shall improve this result and show that the convergence rate is in fact of order α. The proof presented here is actually simpler than that in [12].

We introduce the following definitions.

Definition 2.1. *Let $L \geq 2$. A symbol $q(z,.)$ bounded w.r.t. z with values in $S^1(\mathbb{R}^p \times \mathbb{R}^r)$ is said to satisfy Property (P_L) if it is non-negative and satisfies*

$$(P_L) \quad |\partial_y^{\alpha}\partial_{\eta}^{\beta} q(z,y,\eta)| \leq C(1+|\eta|)^{-|\beta|+(|\alpha|+|\beta|)/L}\,(1+q(z,y,\eta))^{1-(|\alpha|+|\beta|)/L},$$
$$z \in [0,Z],\ y \in \mathbb{R}^p,\ \eta \in \mathbb{R}^r.$$

We then set $\rho = 1 - 1/L$ and $\delta = 1/L$.

Definition 2.2. *Let $L \geq 2$. Let $\rho_\Delta(z,y,\eta)$ be a function in $\mathscr{C}^\infty(\mathbb{R}^p \times \mathbb{R}^r)$ depending on the parameters $\Delta \geq 0$ and $z \in [0,Z]$. We say that ρ_Δ satisfies Property (Q_L) if the following holds*

$$(Q_L) \quad \partial_y^{\alpha}\partial_{\eta}^{\beta}(\rho_\Delta - \rho_\Delta|_{\Delta=0})(z,y,\eta) = \Delta^{m+\delta(|\alpha|+|\beta|)}\rho_\Delta^{m\alpha\beta}(z,y,\eta),$$
$$\text{for } |\alpha| + |\beta| \leq L, \quad 0 \leq m \leq 1 - \delta(|\alpha|+|\beta|),$$

where $\rho_\Delta^{m\alpha\beta}(z,y,\eta)$ is bounded w.r.t. Δ and z with values in $S_\rho^{m-\rho|\beta|+\delta|\alpha|}(\mathbb{R}^p \times \mathbb{R}^r)$. It follows that $\rho_\Delta(z,y,\eta) - \rho_\Delta|_{\Delta=0}(z,y,\eta)$ is itself bounded w.r.t. Δ and z with values in $S_\rho^0(\mathbb{R}^p \times \mathbb{R}^r)$.

We have the following two lemmas [12]

Lemma 2.3. *Let $q(z,y,\eta)$ be bounded w.r.t. z with values in $S^1(\mathbb{R}^p \times \mathbb{R}^r)$. If $q \geq 0$ then q satisfies Property (P_L) for $L = 2$.*

Examples of symbols with such a property with $L > 2$ are given in [19].

Lemma 2.4. *Let $q(z,.)$ be bounded w.r.t. z with values in $S^1(\mathbb{R}^p \times \mathbb{R}^r)$ and satisfy Property (P_L). Define $\rho_\Delta(z,y,\eta) = \exp[-\Delta q(z,y,\eta)]$. Then ρ_Δ satisfies Property (Q_L) for $\Delta \in [0,\Delta_{max}]$ for any $\Delta_{max} > 0$. As $\rho_\Delta|_{\Delta=0} = 1$, ρ_Δ is itself bounded w.r.t. Δ and z with values in $S_\rho^0(\mathbb{R}^p \times \mathbb{R}^r)$.*

We shall assume that c_1 satisfies property (P_L) for some $L \geq 2$. We know that it is always true for $L = 2$ by Lemma 2.3 but special choices for c_1 can be made. Setting

$$p_\Delta(z,x,\xi) := \exp[-\Delta c_1(z,x,\xi)],$$

we obtain that $p_\Delta(z,x,\xi)$ satisfies property (Q_L) by Lemma 2.4.

In the spirit of some of the properties proved in [12] we have

Proposition 2.5. *Let $q_\Delta(z,x,\xi)$ be an amplitude in $S^0_\rho(\mathbb{R}^p \times \mathbb{R}^p)$ depending on the parameters $\Delta \geq 0$ and $z \in [0, Z]$ that satisfies Property (Q_L) for $|\alpha| = 1$ and such that $q_\Delta(z, .)|_{\Delta=0}$ is independent of x. Let $r_z(x,\xi)$ be bounded w.r.t. z with values in $S^s(\mathbb{R}^p \times \mathbb{R}^p)$ for some $s \in \mathbb{R}$. Then*

$$(r_z \# q_\Delta)(z,x,\xi) - r_z(x,\xi)\, q_\Delta(z,x,\xi) = \Delta^{m+\delta} \lambda^m_\Delta(z,x,\xi), \;\; 0 \leq m \leq \rho,$$

where the function $\lambda^m_\Delta(z,x,\xi)$ is bounded w.r.t. Δ and z with values in $S^{m+s-\rho}_\rho(\mathbb{R}^p \times \mathbb{R}^p)$.

Proof. For the sake of concision we take $p = 1$ in the proof, but it naturally extends to $p \geq 1$. Using the oscillatory integral representation of $r_z \# q_\Delta$ we obtain

$$\begin{aligned}
&(r_z \# q_\Delta)(z,x,\xi) - r_z(x,\xi)\, q_\Delta(z,x,\xi) \\
&\quad = \iint \exp[-i\langle y|\xi - \eta\rangle]\, r_z(x,\eta)\, q_\Delta(z, x-y, \xi)\, đ\eta\, dy - r_z(x,\xi)\, q_\Delta(z,x,\xi) \\
&\qquad\qquad = \iint \exp[-i\langle y|\xi - \eta\rangle]\, r_z(x,\eta)\, (q_\Delta(z, x-y, \xi) - q_\Delta(z,x,\xi))\, đ\eta\, dy.
\end{aligned}$$

Taylor's formula yields

$$\begin{aligned}
&(r_z \# q_\Delta)(z,x,\xi) - r_z(x,\xi)\, q_\Delta(z,x,\xi) \\
&\qquad = \int_0^1 \iint -y \exp[-i\langle y|\xi - \eta\rangle]\, r_z(x,\eta)\, \partial_2 q_\Delta(z, x-ty, \xi)\, đ\eta\, dy\, dt.
\end{aligned}$$

With an integration by parts we obtain

$$\begin{aligned}
&(r_z \# q_\Delta)(z,x,\xi) - r_z(x,\xi)\, q_\Delta(z,x,\xi) \\
&\qquad = -i \int_0^1 \iint \exp[-i\langle y|\xi - \eta\rangle]\, \partial_2 r_z(x,\eta)\, \partial_2 q_\Delta(z, x-ty, \xi)\, đ\eta\, dy\, dt.
\end{aligned}$$

Using Property (Q_L) we find

$$\begin{aligned}
&(r_z \# q_\Delta)(z,x,\xi) - r_z(x,\xi)\, q_\Delta(z,x,\xi) = \\
&- i\Delta^{m+\delta} \int_0^1 \iint \exp[-i\langle y|\xi - \eta\rangle]\, \partial_2 r_z(x,\eta)\, q^{m10}_\Delta(z, (1-t)x + t(x-y), \xi)\, đ\eta\, dy\, dt \\
&\qquad\qquad = -i\Delta^{m+\delta} (\partial_2 r_z(x,\xi) \# \widetilde{q}^{m10}_\Delta(z,u,x,\xi))|_{u=x},
\end{aligned}$$

where

$$\widetilde{q}^{m10}_\Delta(z,u,x,\xi) = \int_0^1 q^{m10}_\Delta(z, (1-t)u + tx, \xi)\, dt.$$

As $\widetilde{q}^{m10}_\Delta$ is bounded w.r.t. Δ and z with values in $S^{m+\delta}_\rho(\mathbb{R}^{2p} \times \mathbb{R}^p)$ and $\partial_2 r_z$ is bounded w.r.t. z with values in $S^{s-1}(\mathbb{R}^p \times \mathbb{R}^p)$ we obtain the result. ∎

We recall the following result (Theorem 2.5 and the following remark in [9]), which shall be of use below

Proposition 2.6. *Let $Q_{(z',z)}$ be the global FIO with kernel*

$$Q_{(z',z)}(x',x) = \int \exp[i\langle x' - x|\xi\rangle + i\Delta b_1(z,x',\xi)]\, \sigma_Q(z,x',\xi)\, đ\xi,$$

with $\sigma_Q(z,.)$ bounded w.r.t. z with values in $S^m_\rho(X\times\mathbb{R}^n)$, $m\in\mathbb{R}$, $\frac{1}{2}\le\rho\le 1$. Then for all $s\in\mathbb{R}$ there exist $K=K(s,m)\ge 0$, $\Delta_3>0$ such that

$$\|Q_{(z',z)}\|_{(H^{(s)},H^{(s-m)})}\le K\ p(\sigma_Q(z,.)),$$

for all $z\in[0,Z]$, and $0\le\Delta\le\Delta_3$, where $p(.)$ is some appropriately chosen seminorm on $S^m_\rho(X\times\mathbb{R}^n)$.

In preparation of the main result of this section we define

$$\mathcal{A}_{(z',z)}:=\partial_{z'}\circ\mathcal{G}_{(z',z)},\quad \mathcal{B}_{(z',z)}:=a_z(x,D)\circ\mathcal{G}_{(z',z)}.$$

We have

$$\mathcal{A}_{(z',z)}u(x')=-\iint\exp[i\langle x'-x|\xi\rangle+i\Delta b_1(z,x',\xi)]\\ \times a_z(x',\xi)\ p_\Delta(z,x',\xi)\ g_{(z',z)}(x',\xi)\ u(x)\ dx\ đ\xi.$$

For Δ sufficiently small, the real phase function

$$\varphi_{(z',z)}(x',x,\xi)=\langle x'-x|\xi\rangle+\Delta b_1(z,x',\xi)$$

satisfies Definition 1.2 in [10, Section 10.1] and we have

$$\mathcal{B}_{(z',z)}u(x')=\iint\exp[i\langle x'-x|\xi\rangle+i\Delta b_1(z,x',\xi)]\ \sigma_\Delta(z,x',\xi)\ u(x)\ dx\ đ\xi,$$

from Theorem 2.2 in [10, Section 10.2], where $\sigma_\Delta(z,x,\xi)$ is a symbol in $S^1_\rho(\mathbb{R}^n\times\mathbb{R}^n)$ given, as an oscillatory integral, by

$$\sigma_\Delta(z,x,\xi)=\iint\exp[-i\langle x-y|\xi-\eta\rangle+i\Delta(b_1(z,y,\xi)-b_1(z,x,\xi))]\\ \times p_\Delta(z,y,\xi)\ a_z(x,\eta)\ g_{(z',z)}(y,\xi)\ đ\eta\ dy.$$

We shall need the following regularity assumption w.r.t. z on the symbol $a_z(x,\xi)$.

Assumption 2.7. *The symbol $a(z,.)$ is assumed to be in $\mathscr{C}^{0,\alpha}([0,Z],S^1(\mathbb{R}^n\times\mathbb{R}^n))$, i.e. Hölder continuous w.r.t. z, with values in $S^1(\mathbb{R}^n\times\mathbb{R}^n)$, in the sense that, for some $0<\alpha<1$*

$$a(z',x,\xi)-a(z,x,\xi)=(z'-z)^\alpha\,\widetilde{a}(z',z,x,\xi),\quad 0\le z\le z'\le Z,$$

with $\widetilde{a}(z',z,x,\xi)$ bounded w.r.t. z' and z with values in $S^1(\mathbb{R}^n\times\mathbb{R}^n)$.

The following result is the key point in the proof of the convergence theorem below.

Theorem 2.8. *Let $s\in\mathbb{R}$. There exist $\Delta_4>0$ and $C\ge 0$ such that for $z'-z=\Delta$, $\Delta\in[0,\Delta_4]$,*

$$\|(\partial_{z'}+a_{z'}(x,D_x))\circ\mathcal{G}_{(z',z)}\|_{(H^{(s)},H^{(s-1)})}\le C\Delta^\alpha.$$

Proof. In the proof, we shall always assume that Δ is sufficiently small to apply the invoked properties and results. We have $\|\mathcal{G}_{(z',z)}\|_{(H^{(s)},H^{(s)})}\le C$ by Theorem 1.5. With Assumption 2.7 we have

$$\|(a_z(x,D_x)-a_{z'}(x,D_x))\circ\mathcal{G}_{(z',z)}\|_{(H^{(s)},H^{(s-1)})}\le C\Delta^\alpha,$$

by Theorem 18.1.13 in [5]. It is thus sufficient to prove

$$\|\mathcal{A}_{(z',z)} + \mathcal{B}_{(z',z)}\|_{(H^{(s)},H^{(s-1)})} = \|(\partial_{z'} + a_z(x, D_x)) \circ \mathcal{G}_{(z',z)}\|_{(H^{(s)},H^{(s-1)})} \le C\Delta.$$

With the first-order Taylor formula and Lemma 18.1.10 in [5] we find

$$g_{(z',z)}(x,\xi) := 1 + \Delta\widetilde{g}_{(z',z)}(x,\xi), \tag{2.1}$$

with $\widetilde{g}_{(z',z)}$ bounded w.r.t. Δ and w.r.t. z with values in $S^0(\mathbb{R}^n \times \mathbb{R}^n)$. We can thus write $\mathcal{A}_{(z',z)} = \underline{\mathcal{A}}_{(z',z)} + \Delta\widetilde{\mathcal{A}}_{(z',z)}$ with

$$\widetilde{\mathcal{A}}_{(z',z)}u(x') = -\iint \exp[i\langle x' - x|\xi\rangle + i\Delta b_1(z, x', \xi)] \\ \times a_z(x',\xi)\, p_\Delta(z, x', \xi)\, \widetilde{g}_{(z',z)}(x',\xi)\, u(x)\, dx\, đ\xi,$$

and $\underline{\mathcal{A}}_{(z',z)}$ is given by the same formula with $\widetilde{g}_{(z',z)}$ replaced by 1. With Proposition 2.6 we find $\|\widetilde{\mathcal{A}}_{(z',z)}\|_{(H^{(s)},H^{(s-1)})} \le C$ since $a_z(x,\xi)\, p_\Delta(z,x,\xi)\widetilde{g}_{(z',z)}(x,\xi)$ is bounded w.r.t. z' and z with values in $S^1_\rho(\mathbb{R}^n \times \mathbb{R}^n)$.

Similarly we write the amplitude of $\mathcal{B}_{(z',z)}$ as

$$\sigma_\Delta(z,x,\xi) = \underline{\sigma}_\Delta(z,x,\xi) + \Delta\widetilde{\sigma}_\Delta(z,x,\xi),$$

where

$$\widetilde{\sigma}_\Delta(z,x,\xi) = \iint \exp[-i\langle x - y|\xi - \eta\rangle + i\Delta(b_1(z,y,\xi) - b_1(z,x,\xi))] \\ \times p_\Delta(z,y,\xi)\, a_z(x,\eta)\, \widetilde{g}_{(z',z)}(y,\xi)\, đ\eta\, dy,$$

and $\underline{\sigma}_\Delta(z,x,\xi)$ is given by

$$\underline{\sigma}_\Delta(z,x,\xi) = \iint \exp[-i\langle x - y|\xi - \eta\rangle + i\Delta(b_1(z,y,\xi) - b_1(z,x,\xi))] \\ \times p_\Delta(z,y,\xi)\, a_z(x,\eta)\, đ\eta\, dy. \tag{2.2}$$

From Theorem 2.2 in [10, Section 10.2] (and its proof) we find that $\widetilde{\sigma}_\Delta(z,x,\underline{\eta})$ is bounded w.r.t. Δ and z with values in $S^1_\rho(\mathbb{R}^n \times \mathbb{R}^n)$. Writing $\mathcal{B}_{(z',z)} = \underline{\mathcal{B}}_{(z',z)} + \Delta\widetilde{\mathcal{B}}_{(z',z)}$ accordingly, we obtain $\|\widetilde{\mathcal{B}}_{(z',z)}\|_{(H^{(s)},H^{(s-1)})} \le C$ with Proposition 2.6. To conclude, it thus suffices to prove $\|\underline{\mathcal{A}}_{(z',z)} + \underline{\mathcal{B}}_{(z',z)}\|_{(H^{(s)},H^{(s-1)})} \le C\Delta$. By Proposition 2.6, this follows from the next lemma. ∎

We denote by κ_Δ the amplitude of the FIO $\underline{\mathcal{A}}_{(z',z)} + \underline{\mathcal{B}}_{(z',z)}$. We have

Lemma 2.9. *The symbol $\kappa_\Delta(z,x,\xi) := \underline{\sigma}_\Delta(z,x,\xi) - a_z(x,\xi)\, p_\Delta(z,x,\xi)$ can be written as $\Delta\widetilde{\kappa}_\Delta(z,x,\xi)$ with $\widetilde{\kappa}_\Delta(z,x,\xi)$ bounded w.r.t. Δ and z with values in $S^1_\rho(\mathbb{R}^n \times \mathbb{R}^n)$.*

Proof. We first write $\kappa_\Delta(z,x,\xi) = \kappa_{\Delta,1}(z,x,\xi) + \kappa_{\Delta,2}(z,x,\xi)$ with

$$\kappa_{\Delta,1}(z,x,\xi) := \underline{\sigma}_\Delta(z,x,\xi) - (a_z \# p_\Delta(z,.))(x,\xi),$$
$$\kappa_{\Delta,2}(z,x,\xi) := (a_z \# p_\Delta(z,.))(x,\xi) - a_z(x,\xi)\, p_\Delta(z,x,\xi),$$

and work on each one separately.

The composition product of ψDOs gives [5, Theorem 18.1.8]

$$(a_z \,\#\, p_\Delta(z,.))(x,\xi) = \iint \exp[-i\langle x-y|\xi-\eta\rangle]\, a_z(x,\eta)\, p_\Delta(z,y,\xi)\, đ\eta\, dy.$$

We thus obtain

$$\begin{aligned}\kappa_{\Delta,1}(z,x,\xi) = \iint \exp[-i\langle x-y|\xi-\eta\rangle]\ &(\exp[i\Delta v(z,x,y,\xi)]-1)\\ &\times a_z(x,\eta)\, p_\Delta(z,y,\xi)\, đ\eta\, dy,\end{aligned}$$

as an oscillatory integral where

$$v(z,x,y,\xi) = b_1(z,y,\xi) - b_1(z,x,\xi) = \langle y-x|h(z,x,y,\xi)\rangle,$$

for $h(z,x,y,\xi)$ continuous w.r.t. z with values in $S^1(\mathbb{R}^{2n}\times\mathbb{R}^n)$, homogeneous of degree 1 by Assumption 1.1 and estimate (1.1.9) in [7]. Taylor's formula yields

$$\begin{aligned}\kappa_{\Delta,1}(z,x,\xi) &= i\Delta \int_0^1 \iint \exp[-i\langle x-y|\xi-\eta\rangle]\ \exp[is\Delta v(z,x,y,\xi)]\\ &\qquad \times \langle y-x|h(z,x,y,\xi)\rangle\, a_z(x,\eta)\, p_\Delta(z,y,\xi)\, đ\eta\, dy\, ds\\ &= -\Delta \int_0^1 \iint \langle h(z,x,y,\xi)|\nabla_\eta \exp[-i\langle x-y|\xi-\eta\rangle]\rangle\ \exp[is\Delta v(z,x,y,\xi)]\\ &\qquad \times a_z(x,\eta)\, p_\Delta(z,y,\xi)\, đ\eta\, dy\, ds,\end{aligned}$$

which after integration by parts gives

$$\begin{aligned}\kappa_{\Delta,1}(z,x,\xi) = \Delta \int_0^1 \iint \exp[-i\langle x-y|\xi-\eta\rangle]\ &\exp[is\Delta(b_1(z,y,\xi)-b_1(z,x,\xi))]\\ &\langle h(z,t,y,\xi)|\nabla_\eta a_z(x,\eta)\rangle\, p_\Delta(z,y,\xi)\, đ\eta\, dy\, ds|_{t=x}.\end{aligned}$$

We thus recover a composition formula for a ΨDO and an FIO, as that given in Theorem 2.2 in [10, Section 10.2], with t, s and z as parameters. For $j = 1,\dots,n$, $h_j(z,t,y,\xi)p_\Delta(z,y,\xi)$ is continuous w.r.t. z and smooth and bounded w.r.t. $t\in\mathbb{R}^n$ with values in $S^1_\rho(\mathbb{R}^n\times\mathbb{R}^n)$ and $\partial_{\eta_j} a_z(x,\eta)$ is continuous w.r.t. z with values in $S^0(\mathbb{R}^n\times\mathbb{R}^n)$. We thus obtain that $\kappa_{\Delta,1}(z,x,\xi) = \Delta\widetilde{\kappa}_{\Delta,1}(z,x,\xi)$ with $\widetilde{\kappa}_{\Delta,1}(z,x,\xi)$ continuous w.r.t. z and bounded w.r.t. Δ with values in $S^1_\rho(\mathbb{R}^n\times\mathbb{R}^n)$.

For the second term, $\kappa_{\Delta,2}(z,x,\xi)$, we apply Proposition 2.5 with $m=\rho$ and obtain $\kappa_{\Delta,2}(z,x,\xi) = \Delta\widetilde{\kappa}_{\Delta,2}(z,x,\xi)$ with $\widetilde{\kappa}_{\Delta,2}(z,x,\xi)$ bounded w.r.t. z and Δ with values in $S^1_\rho(\mathbb{R}^n\times\mathbb{R}^n)$. ■

With the same line of arguments as in [12, Section 3] we obtain, from Theorem 2.8, the following convergence result

Theorem 2.10. *Assume that $a(z,.)$ is in $\mathscr{C}^{0,\alpha}([0,Z],S^1(\mathbb{R}^n\times\mathbb{R}^n))$, i.e. Hölder continuous w.r.t. z, with values in $S^1(\mathbb{R}^n\times\mathbb{R}^n)$, in the sense that, for some $0<\alpha<1$*

$$a(z',x,\xi) - a(z,x,\xi) = (z'-z)^\alpha\, \widetilde{a}(z',z,x,\xi),\quad 0\le z\le z'\le Z,$$

or Lipschitz ($\alpha = 1$), with $\widetilde{a}(z', z, x, \xi)$ bounded w.r.t. z' and z with values in $S^1(\mathbb{R}^n \times \mathbb{R}^n)$. Let $s \in \mathbb{R}$ and $0 \leq r < 1$. Then the approximation Ansatz $\mathcal{W}_{\mathfrak{P},z}$ converges to the solution operator $U(z, 0)$ of the Cauchy problem (1.1)–(1.2) in $L(H^{(s+1)}(\mathbb{R}^n), H^{(s+r)}(\mathbb{R}^n))$ uniformly w.r.t. z as $\Delta_{\mathfrak{P}}$ goes to 0 with a convergence rate of order $\alpha(1 - r)$:

$$\|\mathcal{W}_{\mathfrak{P},z} - U(z, 0)\|_{(H^{(s+1)},H^{(s+r)})} \leq C\Delta_{\mathfrak{P}}^{\alpha(1-r)}, \quad z \in [0, Z].$$

Furthermore, the operator $\mathcal{W}_{\mathfrak{P},z}$ strongly converges to the solution operator $U(z, 0)$ in $L(H^{(s+1)}(\mathbb{R}^n), H^{(s+1)}(\mathbb{R}^n))$ uniformly w.r.t. $z \in [0, Z]$.

A result similar to that of the previous theorem can be obtained with weaker assumptions on the symbol $a(z, .)$ by introducing another, yet natural, Ansatz to approximate the exact solution to the Cauchy problem (1.1)–(1.2). For a symbol $q(z, y, \eta) \in \mathscr{C}^0([0, Z], S^m(\mathbb{R}^p \times \mathbb{R}^r))$ we define $\widehat{q}_{(z',z)}(y, \eta) \in \mathscr{C}^0([0, Z]^2, S^m(\mathbb{R}^p \times \mathbb{R}^r))$

$$\widehat{q}_{(z',z)}(y, \eta) := \frac{1}{z' - z} \int_z^{z'} q(s, y, \eta)\, ds, \quad 0 \leq z < z' \leq Z.$$

Then we define

$$\begin{aligned}(2.3)\quad \widehat{\phi}_{(z',z)}(x', x, \xi) &:= \langle x' - x|\xi\rangle + i\Delta\widehat{a}_{1(z',z)}(x', \xi) \\ &= \langle x' - x|\xi\rangle + \Delta\widehat{b}_{1(z',z)}(x', \xi) + i\Delta\widehat{c}_{1(z',z)}(x', \xi).\end{aligned}$$

and

$$(2.4) \qquad \widehat{g}_{(z',z)}(x, \xi) := \exp[-\Delta\widehat{a}_{0(z',z)}(x, \xi)]$$

and finally, following [11], we denote by $\widehat{\mathcal{G}}_{(z',z)}$ the FIO with distribution kernel

$$\begin{aligned}\widehat{G}_{(z',z)}(x', x) &= \int \exp[i\langle x' - x|\xi\rangle] \exp[-\Delta\widehat{a}_{(z',z)}(x', \xi)]\, đ\xi \\ &= \int \exp[i\widehat{\phi}_{(z',z)}(x', x, \xi)]\, \widehat{g}_{(z',z)}(x', \xi)\, đ\xi.\end{aligned}$$

with the associated approximation Ansatz.

Definition 2.11. *Let $\mathfrak{P} = \{z^{(0)}, z^{(1)}, \ldots, z^{(N)}\}$ be a subdivision of $[0, Z]$ with $0 = z^{(0)} < z^{(1)} < \cdots < z^{(N)} = Z$ such that $z^{(i+1)} - z^{(i)} = \Delta_{\mathfrak{P}}$. The operator $\widehat{\mathcal{W}}_{\mathfrak{P},z}$ is defined as*

$$\widehat{\mathcal{W}}_{\mathfrak{P},z} := \begin{cases} \widehat{\mathcal{G}}_{(z,0)} & \text{if } 0 \leq z \leq z^{(1)}, \\ \widehat{\mathcal{G}}_{(z,z^{(k)})} \prod_{i=k}^{1} \widehat{\mathcal{G}}_{(z^{(i)},z^{(i-1)})} & \text{if } z^{(k)} \leq z \leq z^{(k+1)}. \end{cases}$$

Theorem 2.12. *With the sole assumption of the continuity of the symbol $a(z, .)$ w.r.t. z with values in $S^1(\mathbb{R}^n \times \mathbb{R}^n)$ (Assumption 1.1) the same results in as Theorem 2.10 hold for the operator $\widehat{\mathcal{W}}_{\mathfrak{P},z}$, with a convergence rate of order $1 - r$ for the operator convergence in $L(H^{(s+1)}(\mathbb{R}^n), H^{(s+r)}(\mathbb{R}^n))$.*

3 Approximation of the symbol $a_z(x,\xi)$ and consequences

In applications, as shown for instance in [18] and in Appendix A of [12], the symbol of the operator $a(z,x,D_x)$ is often given by an asymptotic series, say

$$a(z,x,\xi) \sim \sum_{j=0}^{\infty} a_{1-j}(z,x,\xi)$$

with $a_{1-j}(z,x,\xi) \in \mathscr{C}^0([0,Z], S^{1-j}(\mathbb{R}^n \times \mathbb{R}^n))$. Here, we shall assume that the symbols $a_{1-j}(z,x,\xi)$, $j \in \mathbb{N}$, are in $\mathscr{C}^{0,\alpha}([0,Z], S^{1-j}(\mathbb{R}^n \times \mathbb{R}^n))$, i.e. Hölder continuous w.r.t. z with values in $S^1(\mathbb{R}^n \times \mathbb{R}^n)$, in the sense that, for some $0 < \alpha \le 1$

$$a_{1-j}(z',x,\xi) - a_{1-j}(z,x,\xi) = (z'-z)^\alpha\, \widetilde{a}_{1-j}(z',z,x,\xi), \quad 0 \le z \le z' \le Z, \;\; j \in \mathbb{N},$$

with $\widetilde{a}_{1-j}(z',z,x,\xi)$ bounded w.r.t. z' and z with values in $S^{1-j}(\mathbb{R}^n \times \mathbb{R}^n)$.

The following lemma show that the regularity of the a_{1-j}'s w.r.t. z can in fact be inherited by their asymptotic series.

Lemma 3.1. *The symbol $a(z,x,\xi) \sim \sum_{j=0}^{\infty} a_{1-j}(z,x,\xi)$ can be chosen, in its class modulo $S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n)$, to be in $\mathscr{C}^{0,\alpha}([0,Z], S^{1-j}(\mathbb{R}^n \times \mathbb{R}^n))$.*

The proof can be made along the line of that of Proposition 18.1.3 in [5].

In practical applications, see e.g. [2, 13, 14], the full symbol is not computed. Instead, one truncates the asymptotic series and uses

$$\underline{a}(z,x,\xi) = \sum_{j=0}^{k} a_{1-j}(z,x,\xi),$$

for some $k \ge 0$. The purpose of this section is to study the consequences of such an approximation of the symbol $a(z,x,\xi)$ for the representation of the solution to the Cauchy problem (1.1)–(1.2) by $\mathcal{W}_{\mathfrak{P},z}$, and its limit, as $\Delta_{\mathfrak{P}}$ goes to zero.

More generally, we consider that for some $k \ge 0$, we have $a - \underline{a} \in \mathscr{C}^{0,\alpha}([0,Z], S^{-k}(\mathbb{R}^n \times \mathbb{R}^n))$. We denote by $\underline{a}_1$ the principal part of $\underline{a}$ and by $\underline{a}_0$ the remaining part. We see that $\underline{a}_i \in \mathscr{C}^{0,\alpha}([0,Z], S^i(\mathbb{R}^n \times \mathbb{R}^n))$, $i = 0, 1$. We note that $a - \underline{a} = a_0 - \underline{a}_0$ since $\underline{a}_1 = a_1$.

We define the distribution kernel in $\mathscr{D}'(X' \times X)$

$$\begin{aligned}\underline{G}_{(z',z)}(x',x) &:= \int \exp[i\langle x'-x|\xi\rangle] \exp[-\Delta \underline{a}(z,x',\xi)]\, đ\xi \\ &= \int \exp[i\phi_{(z',z)}(x',x,\xi)]\, \underline{g}_{(z',z)}(x',\xi)\, đ\xi,\end{aligned}$$

where

$$\underline{g}_{(z',z)}(x,\xi) := \exp[-\Delta \underline{a}_0(z,x,\xi)],$$

and the phase function $\phi_{(z',z)}(x',x,\xi)$ defined as in (1.4).

We denote by $\underline{\mathcal{G}}_{(z',z)}$ the operator with $\underline{G}_{(z',z)}$ for distribution kernel. The operator $\underline{\mathcal{G}}_{(z',z)}$ is a global Fourier integral operator with complex phases by Proposition 1.4 for Δ sufficiently small. We have

Lemma 3.2. *Let $s \in \mathbb{R}$. There exist $\Delta_5 > 0$ and $C > 0$ such that for $z', z \in [0, Z]$ with $z' \geq z$ and $0 \leq \Delta = z' - z \leq \Delta_5$, we have*

$$\|\mathcal{G}_{(z',z)} - \underline{\mathcal{G}}_{(z',z)}\|_{(H^{(s)},H^{(s+k)})} \leq C\Delta\, p(a_z - \underline{a}_z),$$

for some appropriately chosen seminorm p on $S^{-k}(\mathbb{R}^n \times \mathbb{R}^n)$.

Proof. With Taylor's formula we write

$$\sigma_\Delta(z, x, \xi) := \underline{g}_{(z',z)}(x, \xi) - g_{(z',z)}(x', \xi) = \Delta(a_0(z, x, \xi) - \underline{a}_0(z, x, \xi))$$
$$\exp[-\Delta a_0(z, x, \xi)] \int_0^1 \exp[s\Delta(a_0(z, x, \xi) - \underline{a}_0(z, x, \xi))]\, ds.$$

Hence, $\sigma_\Delta(z, x, \xi) = \Delta\widetilde{\sigma}_\Delta(z, x, \xi)$ and observing that

$$\exp[-\Delta a_0(z, x, \xi)] \int_0^1 \exp[s\Delta(a_0(z, x, \xi) - \underline{a}_0(z, x, \xi))]\, ds$$

is bounded w.r.t. z and Δ with values in $S^0(\mathbb{R}^n \times \mathbb{R}^n)$ by Lemma 18.1.10 in [5], the symbol $\widetilde{\sigma}_\Delta(z, .)$ is bounded w.r.t. z and Δ with values in $S^{-k}(\mathbb{R}^n \times \mathbb{R}^n)$. We thus obtain

$$\underline{G}_{(z',z)} - G_{(z',z)} = \Delta \int \exp[i\phi_{(z',z)}(x', x, \xi)]\, \widetilde{\sigma}_\Delta(z, x', \xi)\, đ\xi.$$

We conclude with Proposition 2.26 in [12] (one can also invoke Proposition 2.6 above and consider the previous kernel with an amplitude in $S^{-k}_\rho(\mathbb{R}^n \times \mathbb{R}^n)$, $\frac{1}{2} \leq \rho \leq 1$, and a real phase). ∎

Definition 3.3. *For $z'' \geq z' \geq z \in [0, Z]$ we write $\mathcal{G}_{(z'',z',z)} := \mathcal{G}_{(z'',z')} \circ \mathcal{G}_{(z',z)}$ and more generally for $z^{(N)} \geq z^{(N-1)} \geq \cdots \geq z^{(0)} \in [0, Z]$ we write $\mathcal{G}_{(z^{(N)},\ldots,z^{(0)})} := \mathcal{G}_{(z^{(N)},z^{(N-1)})} \circ \cdots \circ \mathcal{G}_{(z^{(1)},z^{(0)})}$. Similarly we write $\underline{\mathcal{G}}_{(z^{(N)},\ldots,z^{(0)})} := \underline{\mathcal{G}}_{(z^{(N)},z^{(N-1)})} \circ \cdots \circ \underline{\mathcal{G}}_{(z^{(1)},z^{(0)})}$.*

Recall that the thin-slab propagator $\mathcal{G}_{(z',z)}$ is however not a semigroup nor an evolution family here (see [12, Section 3] and [15]). Thus $\mathcal{G}_{(z^{(N)},\ldots,z^{(0)})} \neq \mathcal{G}_{(z^{(N)},z^{(0)})}$.

With the operators we just defined we have

Proposition 3.4. *Let $s \in \mathbb{R}$ and $R \geq 1$. There exist $\Delta_6 > 0$ and $C > 0$ such that for $N \in \mathbb{N}$ with $N\Delta_6 \leq RZ$ and $0 \leq z^{(0)} \leq z^{(1)} \leq \cdots \leq z^{(N)} \leq Z$, with $z^{(j+1)} - z^{(j)} \leq \Delta \leq \Delta_6$, $j = 0, \ldots, N-1$, we have*

$$\|\mathcal{G}_{(z^{(N)},\ldots,z^{(0)})} - \underline{\mathcal{G}}_{(z^{(N)},\ldots,z^{(0)})}\|_{(H^{(s)},H^{(s+k)})} \leq CZ \sup_{z\in[0,Z]} p(a_z - \underline{a}_z) \exp[CZ],$$

for some appropriately chosen seminorm p on $S^{-k}(\mathbb{R}^n \times \mathbb{R}^n)$.

Proof. We write

$$\begin{aligned}
&\mathcal{G}_{(z^{(N)},\ldots,z^{(0)})} - \underline{\mathcal{G}}_{(z^{(N)},\ldots,z^{(0)})} \\
&\quad= \sum_{i=0}^{N-1} \Big(\mathcal{G}_{(z^{(N)},\ldots,z^{(i)})} \circ \underline{\mathcal{G}}_{(z^{(i)},\ldots,z^{(0)})} - \mathcal{G}_{(z^{(N)},\ldots,z^{(i+1)})} \circ \underline{\mathcal{G}}_{(z^{(i+1)},\ldots,z^{(0)})}\Big) \\
&\quad= \sum_{i=0}^{N-1} \mathcal{G}_{(z^{(N)},\ldots,z^{(i+1)})} \circ \Big(\mathcal{G}_{(z^{(i+1)},z^{(i)})} - \underline{\mathcal{G}}_{(z^{(i+1)},z^{(i)})}\Big) \circ \underline{\mathcal{G}}_{(z^{(i)},\ldots,z^{(0)})}.
\end{aligned}$$

From Theorem 1.5, there exists $M > 0$ such that

$$\|\mathcal{G}_{(z^{(N)},\dots,z^{(i+1)})}\|_{(H^{(s+k)},H^{(s+k)})} \leq (1 + M\Delta)^{N-i-1},$$
$$\|\underline{\mathcal{G}}_{(z^{(i)},\dots,z^{(0)})}\|_{(H^{(s)},H^{(s)})} \leq (1 + M\Delta)^{i}.$$

for Δ_6 sufficiently small. Lemma 3.2 then yields

$$\begin{aligned}
&\|\mathcal{G}_{(z^{(N)},\dots,z^{(0)})} - \underline{\mathcal{G}}_{(z^{(N)},\dots,z^{(0)})}\|_{(H^{(s)},H^{(s+k)})} \\
&\qquad\qquad \leq CN\Delta \sup_{z\in[0,Z]} p(a_z - \underline{a}_z)\,(1 + M\Delta)^{N-1} \\
&\qquad\qquad\qquad\qquad \leq CRZ \sup_{z\in[0,Z]} p(a_z - \underline{a}_z)\,(1 + RZM/N)^{N-1}.
\end{aligned}$$

Since the right-hand side converges to $CRZ\ \sup_{z\in[0,Z]} p(a_z - \underline{a}_z)\ \exp[RMZ]$, as an increasing sequence, this concludes the proof. ■

We define the operator $\underline{\mathcal{W}}_{\mathfrak{P},z}$:

Definition 3.5. *Let* $\mathfrak{P} = \{z^{(0)}, z^{(1)}, \dots, z^{(N)}\}$ *be a subdivision of* $[0, Z]$ *with* $0 = z^{(0)} < z^{(1)} < \cdots < z^{(N)} = Z$ *such that* $z^{(i+1)} - z^{(i)} = \Delta_{\mathfrak{P}}$. *The operator* $\underline{\mathcal{W}}_{\mathfrak{P},z}$ *is defined as*

$$\underline{\mathcal{W}}_{\mathfrak{P},z} := \begin{cases} \underline{\mathcal{G}}_{(z,0)} & \text{if } 0 \leq z \leq z^{(1)}, \\ \underline{\mathcal{G}}_{(z,z^{(k)})} \prod_{i=k}^{1} \underline{\mathcal{G}}_{(z^{(i)},z^{(i-1)})} & \text{if } z^{(k)} \leq z \leq z^{(k+1)}. \end{cases}$$

From Proposition 3.4 we have the following theorem which estimates the error in computing $\underline{\mathcal{W}}_{\mathfrak{P},z}(u_0)$ in place of $\mathcal{W}_{\mathfrak{P},z}(u_0)$ and a *convergence result* in $H^{(s+k)}(\mathbb{R}^n)$ if $a_z - \underline{a}_z$ converges to 0 in $S^{-k}(\mathbb{R}^n \times \mathbb{R}^n)$.

Theorem 3.6. *Let* $\mathfrak{P} = \{z^{(0)}, z^{(1)}, \dots, z^{(N)}\}$ *be a subdivision of* $[0, Z]$ *with* $0 = z^{(0)} < z^{(1)} < \cdots < z^{(N)} = Z$ *such that* $z^{(i+1)} - z^{(i)} = \Delta_{\mathfrak{P}}$. *For* $\Delta_{\mathfrak{P}} \leq \Delta_6$ *then*

$$\|\mathcal{W}_{\mathfrak{P},z} - \underline{\mathcal{W}}_{\mathfrak{P},z}\|_{(H^{(s)},H^{(s+k)})} \leq CZ \sup_{z\in[0,Z]} p(a_z - \underline{a}_z)\ \exp[CZ], \quad z \in [0, Z].$$

for some appropriately chosen seminorm p *on* $S^{-k}(\mathbb{R}^n \times \mathbb{R}^n)$.

From Theorem 2.10, we know that the Ansatz $\mathcal{W}_{\mathfrak{P},z}$ strongly converges in $H^{(s)}(\mathbb{R}^n)$ to the solution operator of the Cauchy problem

$$\partial_z u + \underline{a}(z, x, D_x)u = 0, \quad 0 < z \leq Z, \tag{3.1}$$
$$u\,|_{z=0} = u_0 \in H^{(s)}(\mathbb{R}^n). \tag{3.2}$$

and similarly the Ansatz $\underline{\mathcal{W}}_{\mathfrak{P},z}$ strongly converges in $H^{(s)}(\mathbb{R}^n)$ to the solution operator to the Cauchy problem

$$\partial_z \underline{u} + \underline{a}(z, x, D_x)\underline{u} = 0, \quad 0 < z \leq Z, \tag{3.3}$$
$$\underline{u}\,|_{z=0} = u_0 \in H^{(s)}(\mathbb{R}^n). \tag{3.4}$$

With the hypothesis made on the symbol $a(z, x, \xi)$ there exist a unique solution in $\mathscr{C}^0([0, Z], H^{(s)}(\mathbb{R}^n)) \cap \mathscr{C}^1([0, Z], H^{(s-1)}(\mathbb{R}^n))$ to (3.3)–(3.4). To support the 'optimality' of the result of Theorem 3.6 we note that

Proposition 3.7. *There exist $\lambda_0 > 0$ and $C > 0$ such that*

$$\|u(z,.) - \underline{u}(z,.)\|_{H^{(s+k)}} \leq CZ \sup_{z\in[0,Z]} p(a_z - \underline{a}_z)\ \exp[\lambda Z]\|u_0\|_{H^{(s)}}, \quad z \in [0,Z], \lambda > \lambda_0,$$

for some appropriately chosen seminorm p on $S^{-k}(\mathbb{R}^n \times \mathbb{R}^n)$.

Proof. We note that

$$(\partial_z + a_z(x,D_x))(\underline{u} - u) = (\partial_z + a_z(x,D_x))\underline{u} = (a_z(x,D_x) - \underline{a}_z(x,D_x))\underline{u}.$$

With $u_0 \in H^{(s)}(\mathbb{R}^n)$ then $\underline{u}(z,.) \in \mathscr{C}^0([0,Z], H^{(s)}(\mathbb{R}^n)) \cap \mathscr{C}^1([0,Z], H^{(s-1)}(\mathbb{R}^n))$ by Theorem 23.1.2 in [5]. With $a_z(x,\xi) - \underline{a}_z(x,\xi)$ continuous w.r.t. z with values in $S^{-k}(\mathbb{R}^n)$ we obtain $(a_z(x,D_x) - \underline{a}_z(x,D_x))\underline{u} \in \mathscr{C}^0([0,Z], H^{(s+k)})$ and

$$\|(a_z(x,D_x) - \underline{a}_z(x,D_x))\underline{u}\|_{H^{(s+k)}} \leq \sup_{z\in[0,Z]} p(a_z - \underline{a}_z)\ \|\underline{u}\|_{H^{(s)}}$$

for some appropriately chosen seminorm p on $S^{-k}(\mathbb{R}^n \times \mathbb{R}^n)$ by Proposition 2.6. Theorem 23.1.2 in [5] yields that $u - \underline{u} \in \mathscr{C}^0([0,Z], H^{(s+k)}(\mathbb{R}^n))$ and satisfies the energy estimate

$$\begin{aligned}
&\sup_{z\in[0,Z]} \exp[-\lambda z]\ \|u(z,.) - \underline{u}(z,.)\|_{H^{(s+k)}} \\
&\qquad \leq 2\int_0^Z \exp[-\lambda z]\ \|(a_z(x,D_x) - \underline{a}_z(x,D_x))\underline{u}\|_{H^{(s+k)}} dz \\
&\qquad \leq C \sup_{z\in[0,Z]} p(a_z - \underline{a}_z) \int_0^Z \exp[-\lambda z]\ \|\underline{u}\|_{H^{(s)}}\ dz \leq CZ \sup_{z\in[0,Z]} p(a_z - \underline{a}_z)\ \|u_0\|_{H^{(s)}},
\end{aligned}$$

for λ greater than some $\lambda_0 > 0$ which solely depends on s and k since

$$\sup_{z\in[0,Z]} \exp[-\lambda z]\ \|\underline{u}(z,.)\|_{H^{(s)}} \leq \|u_0\|_{H^{(s)}},$$

by Theorem 23.1.2 in [5]. ∎

Remark 3.8. With the result of the previous proposition, one can consider to use a regularized (w.r.t. z) version of $a_z(x,\xi)$, e.g. by convolution with some molifier, say $a_z^\varepsilon(x,\xi)$ converging to $a_z(x,\xi)$ in $\mathscr{C}^{0,\alpha}([0,Z], S^1(\mathbb{R}^n \times \mathbb{R}^n)$ as ε goes to zero. Then the solution u^ε to

$$\partial_z u^\varepsilon + a_z^\varepsilon(x,D_x)u^\varepsilon = 0, \quad 0 < z \leq Z, \tag{3.5}$$

$$u^\varepsilon\ |_{z=0} = u_0 \in H^{(s)}(\mathbb{R}^n), \tag{3.6}$$

converges to the solution u to (3.1)–(3.2) in $H^{(s-1)}$ by Proposition 3.7. The symbol $a_z^\varepsilon(x,\xi)$ is smooth and one can then use the classical FIO parametrix construction to represent u^ε. Such a representation of the solution operator has the drawback of being up to a regularizing operator.

In applications, one is not only interested in the convergence of the Ansatz, be it $\mathcal{W}_{\mathfrak{P},z}(u_0)$ or $\underline{\mathcal{W}}_{\mathfrak{P},z}(u_0)$, to the exact solution of the Cauchy problem (1.1)–(1.2) but one also expect the wavefront set of the approximate solution to be close, in some sense, to that of the exact solution. This is particularly true in applications such as seismology where the wavefront set of the solution is to be connected to interfaces in the subsurface [1].

The propagation of singularities for the proposed Ansatz $\underline{\mathcal{W}}_{\mathfrak{P},z}(u_0)$ is fully govern by the principal part of the symbol $\underline{a}_z(x,\xi)$: following the analysis of [15] we find the following convergence for the wavefront set of $\underline{\mathcal{W}}_{\mathfrak{P},z}(u_0)$, independent of the lower-order symbols that compose $\underline{a}_z$.

Theorem 3.9. *Let $u_0(.) \in H^{(-\infty)}(\mathbb{R}^n)$ and $u(z,.)$, $z \in [0,Z]$, be the solution to the Cauchy problem (1.1)–(1.2). Let $Z' \in [0,Z]$ and K be a compact set in $T^*(\mathbb{R}^n)$ such that for all $\gamma^{(0)} = (x^{(0)}, \xi^{(0)}) \in K \setminus 0$ the bicharacteristics $\chi_z(\gamma^{(0)})$ associated to $-b_1 = Re(a_1)$ originating from $\gamma^{(0)}$ at $z = 0$ satisfies $\chi_z(\gamma^{(0)}) \in \Omega_z$ for all $z \in [0,Z']$ with*

$$\Omega_z = \{(x,\xi) \in T^*(\mathbb{R}^n) \setminus 0;\ (z,x,\xi) \notin \operatorname{supp}(c_1)\}.$$

If $\gamma^{(0)} \in K \cap \mathrm{WF}(u_0)$ we have $\chi_{Z'}(\gamma^{(0)}) \in \mathrm{WF}(u(Z',.))$. For a subdivision $\mathfrak{P}$ of $[0,Z]$, with $\Delta_{\mathfrak{P}}$ sufficiently small, we then have

$$\operatorname{dist}(\chi_z(\gamma^{(0)}), \mathrm{WF}(\underline{\mathcal{W}}_{\mathfrak{P},z}(u_0))) \to 0, \text{ as } \Delta_{\mathfrak{P}} \to 0$$

uniformly w.r.t. $\gamma^{(0)} \in K \cap \mathrm{WF}(u_0)$ and $z \in [0,Z']$. Furthermore, the convergence is of order α, $0 < \alpha \leq 1$, if $b_1(z,.)$ is in $\mathscr{C}^{0,\alpha}([0,Z], S^1(\mathbb{R}^n,\mathbb{R}^n))$.

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References

[1] M. V. de Hoop, J. Le Rousseau, and B. Biondi. Symplectic structure of wave-equation imaging: A path-integral approach based on the double-square-root equation. *Geoph. J. Int.*, 153:52–74, 2003.

[2] M. V. de Hoop, J. Le Rousseau, and R.-S. Wu. Generalization of the phase-screen approximation for the scattering of acoustic waves. *Wave Motion*, 31:43–70, 2000.

[3] J. J. Duistermaat. *Fourier integral operators*. Birkhäuser, Boston, 1996.

[4] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*. Springer-Verlag, Berlin, 1999.

[5] L. Hörmander. *The analysis of linear partial differential operators*, volume III. Springer-Verlag, 1985. Second printing 1994.

[6] L. Hörmander. *The analysis of linear partial differential operators*, volume IV. Springer-Verlag, 1985.

[7] L. Hörmander. *The analysis of linear partial differential operators*, volume I. Springer-Verlag, second edition, 1990.

[8] T. Kato. Linear evolution equations of "hyperbolic" type. *J. Fac. Sci. Univ. Tokyo, Sec. I*, 17:241–258, 1970.

[9] H. Kumano-go. A calculus of Fourier integral operators on $\mathbb{R}^N$ and the fundamental solution for an operator of hyperbolic type. *Comm. Part. Diff. Eqs.*, 1(1):1–44, 1976.

[10] H. Kumano-go. *Pseudo-differential operators*. MIT Press, Cambridge, 1981.

[11] N. Kumano-go. A construction of the fundamental solution for Schrödinger equations. *J. Math. Sci. Univ. Tokyo*, 2:441–498, 1995.

[12] J. Le Rousseau. Fourier-integral-operator approximation of solutions to pseudodifferential first-order hyperbolic equations I: convergence in Sobolev spaces. *Comm. Part. Diff. Eqs.*, 31:867–906, 2006.

[13] J. Le Rousseau and M. V. de Hoop. Modeling and imaging with the scalar generalized-screen algorithms in isotropic media. *Geophysics*, 66:1551–1568, 2001.

[14] J. Le Rousseau and M. V. de Hoop. Generalized-screen approximation and algorithm for the scattering of elastic waves. *Q. J. Mech. Appl. Math.*, pages 1–33, 2003.

[15] J. Le Rousseau and G. Hörmann. Fourier-integral-operator approximation of solutions to pseudodifferential first-order hyperbolic equations II: microlocal analysis. *J. Math. Pures Appl.*, 86:403–426, 2006.

[16] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*. Springer-Verlag, New York, 1983.

[17] H. Smith. A parametrix construction for wave equations with $C^{1,1}$ coefficients. *Ann. Inst. Fourier*, 48:797–835, 1998.

[18] C. C. Stolk. A pseudodifferential equation with damping for one-way wave propagation in inhomogeneous media. *Wave Motion*, 40(2):111–121, 2004.

[19] C. C. Stolk. Parametrix for a hyperbolic intial value problem with dissipation in some region. *Asymptotic Analysis*, 43(1-2):151–169, 2005.